

2PN/RM gauge invariance in Brans-Dicke-like scalar-tensor theories

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In this note we study the 2PN/RM gauge invariance structure of a *Brans-Dicke-like* Scalar-Tensor Theories (STT) without potential. Since the spherical isotropic metric plays an important role in the literature, its 2PN/RM STT version is deduced from the general equations given in [10], by using the invariance structure properties. It is found that the second order Eddington parameter ϵ can be written in terms of the usual post-Newtonian parameter γ and β as $\epsilon = 4/3\gamma^2 + 4/3\beta - 1/6\gamma - 3/2$.

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I. INTRODUCTION

Because of their expected accuracies, most projects relying on precise measurements of the characteristics of the propagation of light in the solar-system require the knowledge of the space-time metric at the c^{-4} level ¹ in order to have sufficiently accurate equations describing the various observables (see for instance [4, 8, 10] and references therein).

In a recent work [10] give the 2PN/RM metric of Scalar-Tensor Theories (STT) without potential, in a set of coordinates that respect the Weak Spatial Isotropy Condition (WSIC: $g_{ij} \propto \delta_{ij} + O(c^{-4})$, [3]); otherwise arbitrary. Although eventually such kind of dynamical metric would have to be taken into account for accurate solar system calculations, a lot of works exploring the 2PN/RM phenomenology simplify the problem by considering the spherical case in isotropic coordinates [1, 5, 7, 11, 12, 14, 15]. This allows to explore the main features of the 2PN/RM phenomenology; without the complication of a more realistic dynamical metric as given for instance by [10]. Hence, a parameterized 2PN/RM metric is often considered, with a new parameter ϵ entering in the c^{-4} space-space part of the metric.

In this note, we first study the gauge invariance structure left in the 2PN/RM field equations, which are such that the coordinate system respects the WSIC only. Then we use this analysis in order to derive the value of the second order post-Newtonian parameter ϵ in the case of STT without potential. This last calculation illustrates the method that uses the gauge invariant structure of the field equations in order to find specific gauges.

II. FIELD EQUATIONS

We start with the usual STT action without potential in the Jordan representation [10]:

$$S = \frac{c^4}{16\pi G} \int d^4x \sqrt{-g} \left[\Phi R - \frac{\omega(\Phi)}{\Phi} g^{\alpha\beta} \partial_\alpha \Phi \partial_\beta \Phi \right] + \int d^4x \sqrt{-g} L_{NG}(\Psi, g_{\mu\nu}). \quad (1)$$

g is the metric determinant, R is the Ricci scalar constructed from the *physical* metric $g_{\mu\nu}$ ², \mathcal{L}_m is the material Lagrangian, Ψ represents the non-gravitational fields. From there, one deduces the following 2PN/RM metric as well

¹ The c^{-4} metric is also called the 2PN/RM metric; where PN/RM stands for Post Newtonian/Relativistic Motion. It means that the development of the Post-Newtonian metric is developed to the order that has to be taken into account when dealing with test particles with relativistic velocities only [9, 10].

² In our model, we assume that $g_{\mu\nu}$ is the *physical* metric, in the sense that it is the one that describes actual time and length as measured by clocks and rods in our experiments [6].

as the corresponding field equations [10]:

$$\begin{aligned} g_{00} &= -1 + \frac{2W}{c^2} - \beta \frac{2W^2}{c^4} + O(c^{-5}), \\ g_{0i} &= -(\gamma + 1) \frac{2W_i}{c^3} + O(c^{-5}), \\ g_{ij} &= \delta_{ij} \left\{ 1 + \gamma \frac{2W}{c^2} + (\gamma^2 + \beta - 1) \frac{2W^2}{c^4} \right\} + (\gamma + 1) \frac{2W_{ij}}{c^4} + O(c^{-5}), \end{aligned} \quad (2)$$

with

$$\gamma \equiv \frac{\omega_0 + 1}{\omega_0 + 2}, \quad \beta \equiv 1 + \frac{\omega'_0}{(2\omega_0 + 3)(2\omega_0 + 4)^2}, \quad G_{eff} \equiv \frac{2\omega_0 + 4}{2\omega_0 + 3} G,$$

and

$$\begin{aligned} \square W + \frac{1 + 2\beta - 3\gamma}{c^2} W \Delta W + \frac{2}{c^2} (1 + \gamma) \partial_t J &= -4\pi G_{eff} \Sigma + O(c^{-3}), \\ \Delta W_i - \partial_i J &= -4\pi G_{eff} \Sigma^i + O(c^{-2}), \\ \Delta W_{ij} + \partial_i W \partial_j W + 2(1 - \beta) \delta_{ij} W \Delta W - \partial_i J_j - \partial_j J_i - 2\gamma \delta_{ij} \partial_t J &= -4\pi G_{eff} \Sigma^{ij} + O(c^{-1}), \end{aligned} \quad (3)$$

Where one has set

$$\begin{aligned} J &= \partial_t W + \partial_k W_k + O(c^{-2}), \\ J_i &= \partial_k W_{ik} - \frac{1}{2} \partial_i W_{kk} + \partial_t W_i - \frac{1 - \gamma}{2} \partial_i P, \end{aligned} \quad (4)$$

with

$$\Delta P + 2 \frac{\beta - 1}{1 - \gamma} W \Delta W - 2 \partial_t J = -4\pi G_{eff} \frac{\Sigma^{kk}}{3\gamma - 1} + O(c^{-1}), \quad (5)$$

while for the matter part of the equations, one has:

$$\Sigma = \frac{1}{c^2} (T^{00} + \gamma T^{kk}), \quad \Sigma^i = \frac{1}{c} T^{0i}, \quad \Sigma^{ij} = T^{ij} - \gamma T^{kk} \delta_{ij}.$$

III. THE GAUGE INVARIANCE STRUCTURE

As in the General Relativity (GR) case, the diffeomorphism invariance leaves 4 degrees of freedom (dof.) left unconstrained in the field equations. Therefore, there is a *gauge-invariance-like* behavior of such field equations. Because of the WSIC ($g_{ij} \propto \delta_{ij} + O(c^{-4})$) imposed to the metric at the 1PN level, the 1PN field equation gauge freedom is characterized by an arbitrary scalar function λ only. Indeed, the WSIC in the Jordan representation follows from the Strong Spatial Isotropy Condition (SSIC: $g_{00}g_{ij} = -\delta_{ij} + O(c^{-4})$, [3]) in the Einstein representation [10]. Therefore, it fixes the gauge freedom corresponding to the spatial dof. that would appear in the field equations otherwise; and leaves only the gauge freedom corresponding the choice of time coordinates. One has:

$$W' = W - \frac{1}{c^2} \partial_t \lambda, \quad W'_i = W_i + \frac{1}{2(1 + \gamma)} \partial_i \lambda. \quad (6)$$

At the 2PN/RM level, the metric in general can't be put in an isotropic form anymore, and the spatial dof. reappear anew in the field equations, as in GR [9]. Therefore, at this level, the gauge invariance is characterized by an additional arbitrary 3-vector A_i , such that:

$$W'_{kl} = W_{kl} + \partial_k A_l + \partial_l A_k + \frac{\gamma}{1 + \gamma} \delta_{kl} \partial_t \lambda. \quad (7)$$

But to be complete, one also has to take into account the invariance of the equation on the scalar field P :

$$P' = P + \frac{1}{1 + \gamma} \partial_t \lambda. \quad (8)$$

This scalar field is a *leftover* of the scalar field Φ in the field equations at the c^{-4} level. It is due to the fact that the scalar field equation's source is proportional to the trace of the stress-energy tensor instead of being proportional to Σ .

Equations (6-8) represent the 2PN/RM gauge invariance structure of the scalar-tensor field equations in a set of coordinates that respect the WSIC.

IV. THE SPHERICAL ISOTROPIC CASE

The spherical isotropic case has an important place in the literature related to the 2PN/RM metric. Indeed, it looks like the simplest metric one can use in order to derive the 2PN/RM phenomenology characterized for instance by the time transfer or deviation angle equations. In this case – corresponding to a spherical source at the center of the coordinates – the metric in various alternative theories would write, according to [5]:

$$g_{00} = -1 + \frac{2W'}{c^2} - \beta \frac{2W'^2}{c^4} + O(c^{-5}), \quad (9)$$

$$g_{0i} = O(c^{-5}), \quad (10)$$

$$g_{ij} = \delta_{ij} \left\{ 1 + \gamma \frac{2W'}{c^2} + \frac{3\epsilon}{2} \frac{W'^2}{c^4} \right\} + O(c^{-5}), \quad (11)$$

with $W' = GM/r'$, where G is the effective gravitational constant, M the mass of the source and r' the radial coordinate in the isotropic system of coordinates. We dub ϵ : second order Eddington parameter – equal to 1 in GR. However, one should notice that there is no reason to expect that a vector-tensor theory for instance, would not break the spherical symmetry of the problem because of the local direction of the space part of the vector field. Therefore, such a metric seems to be useful for a very restricted set of alternative theories – namely, probably only scalar-tensor theories. However, while in our opinion one should use the metric in the general form, such as the one given by equations (2), it still seems interesting to give the value of the ϵ parameter in scalar-tensor theories – mainly because most of the papers studying some aspects of the 2PN/RM phenomenology use the metric in the form given by equations (9-11) [1, 5, 7, 11–15].

In order to get the metric in this class of coordinate system, one has to find a gauge transformation that kills the anisotropic terms in equation (3). This can be achieved by realizing that $\partial_i W \partial_j W = \frac{1}{8} \partial_{ij}^2 W^2 + \delta_{ij} U$, where $W = GM/r - r$ being the original radial coordinate (ie. in no specific coordinate system) – and $U = \frac{1}{4}(GM/r^2)^2$. Therefore, since we are considering a static vacuum field solution where $\Delta W = O(c^{-2})$, $\partial_t J = 0$ and $S_{ij} = 0$, equation (3) writes:

$$\Delta W_{ij} + \frac{1}{8} \partial_{ij}^2 W^2 + \delta_{ij} U - \partial_i J_j - \partial_j J_i = O(c^{-1}), \quad (12)$$

which can be re-written as:

$$\Delta W'_{ij} + \partial_i \left(\Delta A_j + \frac{1}{16} \partial_j W^2 - J_j \right) + \partial_j \left(\Delta A_i + \frac{1}{16} \partial_i W^2 - J_i \right) = -\delta_{ij} U + O(c^{-1}), \quad (13)$$

by using the gauge invariance of the field equation as well as the commutativity of partial derivatives. Then by choosing the 3-vector gauge field A_i that satisfies the following equation:

$$\Delta A_i = J_i - \frac{1}{16} \partial_i W^2 = \partial_k W_{ki} - \frac{1}{2} \partial_i W_{kk} + \partial_t W_i - \frac{1-\gamma}{2} \partial_i P - \frac{1}{16} \partial_i W^2, \quad (14)$$

³ – which is obviously invertible – equation (12) re-writes:

$$\Delta W'_{ij} = -\delta_{ij} \frac{1}{4} \left(\frac{GM}{r'^2} \right)^2 + O(c^{-1}) \rightarrow W'_{ij} = -\delta_{ij} \frac{1}{8} W'^2 + O(c^{-1}), \quad (15)$$

³ On the contrary, if one wants to write the metric in the harmonic gauge [10], one needs to impose: $\Delta A_i = J_i$ as in General Relativity [9].

with $r' = r + O(c^{-2})$. From there, after injecting into the space-space component of the metric (2), one gets:

$$\epsilon = \frac{4}{3}\gamma^2 + \frac{4}{3}\beta - \frac{\gamma}{6} - \frac{3}{2}. \quad (16)$$

It is the result found in [2] after they considered directly the 2PN spherical body solution in isotropic coordinates of the STT field equations (see their equation (5.8)). As a corollary, setting $\gamma = 1$ and $\beta = 1$ in (14) gives the corresponding transformation in the GR case.

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